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Nontrivial Solutions for Resonant Hemivariational Inequalities

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Abstract. We study a resonant semilinear elliptic hemivariational inequality. Under some assumptions of strong resonance on the Clarke subdifferential of the superpotential, and using nonsmooth critical point theory, the existence of a nontrivial solution of the problem is shown.

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1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be bounded domain with a C^1-boundary. In this paper we study the following resonant semilinear elliptic problem with a nonsmooth potential (hemivariational inequality):

$$\begin{aligned} &-\Delta x(z) - \lambda_k x(z) \in \partial j\big(z, x(z)\big) \qquad \text{for a.a. } z \in Z, \\ &x|_{\partial Z} = 0, \end{aligned} \tag{1.1}$$

where λ_k, $k \geqslant 2$, is an eigenvalue of $\big(-\Delta, H_0^1(Z)\big)$. We prove the existence of nontrivial solutions under the assumption that the subdifferential $\partial j(z, \zeta)$ is bounded by an $L^\infty(Z)$ function. So our analysis incorporates the so called "strongly resonant case", according to the terminology of Bartolo et al. [1]. It is well known that in this case the difficulty arises from the lack of compactness, namely the Palais–Smale condition (in this case its nonsmooth variant) does not hold for all $c \in \mathbb{R}$. Moreover, in this case for every $u \in \partial j(z, \zeta)$, we have

$$\frac{u + \lambda_k \zeta}{\zeta} \longrightarrow \lambda_k \quad \text{as } |\zeta| \to +\infty,$$

which means that we have a completely resonant problem.

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In the past, problem (1.1) was investigated primarily in the context of "smooth problems", (i.e. $j(z,\cdot) \in C^1(\mathbb{R})$). We refer to the works of Solimini [2], de Figueiredo and Gossez [3], Capozzi et al. [4], Hirano and Nishimura [5], Iannacci and Nkashama [6] and the references therein. De Figueieredo and Gossez [3] and Iannacci and Nkashama [6] examined the incomplete resonant case. De Figueieredo and Gossez [3] employed density conditions for $j(z,\zeta)$ at $\pm\infty$ with respect to the first eigenvalue, while Iannacci and Nkashama [6] deal with resonance at higher eigenvalues. Solimini [2], Capozzi et al. [4], Hirano and Nishimura [5] consider the strongly resonant problem. In all these works the right-hand side nonlinearity is independent on $z \in Z$ and has restrictive differentiability properties. Hirano and Nishimura [5] prove multiplicity results.

The study of this problem for hemivariational inequalities is lagging behind. There are some recent works of Goeleven et al. [7], Gasiński and Papageorgiou [8]. Goeleven et al. employ certain Landesman–Lazer type condition, suitably adopted to the nonsmooth, multivalued setting provided by hemivariational inequalities. On the other hand, Gasiński and Papageorgiou [8] consider nonlinear problems driven by the p-Laplacian but their analysis does not include the strongly resonant case.

2. Mathematical Background

As we already mentioned our approach is based on the theory of the nonsmooth critical point theory for locally Lipschitz functionals. For the convenience of the reader in this section we present some basic definitions and facts from this theory which we shall need in the sequel.

Let X be a Banach space and X^* its topological dual. By $\|\cdot\|_X$ we denote the norm of X and by $\langle\cdot,\cdot\rangle_X$ the duality pairing for the pair (X, X^*). In our nonsmooth case crucial role play locally Lipschitz functionals.

A function φ: $X \to \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$ there exists a neighbourhood U of x and a constant $k_U > 0$ such that

$$|\varphi(z) - \varphi(y)| \leqslant k_U \|z - y\|_X \quad \forall z, y \in U.$$

From convex analysis it is known that a proper (i.e. not identically $+\infty$), convex and lower semicontinuous function ψ: $X \to \overline{\mathbb{R}} \stackrel{df}{=} \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\mathrm{dom}\psi \stackrel{df}{=} \{x \in X: \psi(x) < +\infty\}$. In analogy with the directional derivative of a convex function, for a locally Lipschitz function φ: $X \to \mathbb{R}$, we introduce the *generalized directional derivative* of φ at $x \in X$ in the direction $h \in X$, defined by

$$\varphi^0(x;h) \stackrel{\mathrm{d}f}{=} \limsup_{\substack{x' \to x \\ t \searrow 0}} \frac{\varphi(x'+th)-\varphi(x')}{t}$$

(see Ref. [9]). If φ is also convex, then $\varphi^0(x;\cdot)=\varphi'(x;\cdot)$, where $\varphi'(x;\cdot)$ is the usual directional derivative at $x\in X$ of the convex function φ. It is easy to check that the function $X\ni h\to\varphi^0(x;h)\in\mathbb{R}$ is sublinear, continuous, so by the Hahn–Banach theorem, $\varphi^0(x;\cdot)$ is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x)$, defined by

$$\partial\varphi(x) \stackrel{\mathrm{d}f}{=} \{x^*\in X^*: \langle x^*,h\rangle_X \leqslant \varphi^0(x;h) \text{ for all } h\in X\}.$$

The multifunction $\partial\varphi: X\to 2^{X^*}\setminus\{\emptyset\}$ is known as the *generalized* (or *Clarke*) *subdifferential* of φ. From convex analysis we know that if $\psi: X\to\mathbb{R}$ is continuous convex (hence locally Lipschitz), its subdifferential in the sense of convex analysis is given by

$$\partial\psi(x) \stackrel{\mathrm{d}f}{=} \{x^*\in X^*: \langle x^*,h\rangle_X \leqslant \psi'(x;h) \text{ for all } h\in X\}.$$

Since $\psi'(x,\cdot)=\psi^0(x,\cdot)$, we see that for continuous convex (hence locally Lipschitz) functions, the convex subdifferential and the Clarke subdifferential coincide. If $\varphi,\psi: X\to\mathbb{R}$ are two locally Lipschitz functions and $t\in\mathbb{R}$, then

$$\partial(\varphi+\psi)(x)\subseteq\partial\varphi(x)+\partial\psi(x) \quad \forall x\in X$$

(with equality if in addition ψ is convex) and

$$\partial(t\varphi)(x)=t\partial\varphi(x) \quad \forall x\in X.$$

If $\varphi\in C^1(X)$, then $\partial\varphi(x)=\{\varphi'(x)\}$. The multifunction $\partial\varphi$ is upper semicontinuous from X into $X^*_{w^*}$ (by $X^*_{w^*}$ we denote the space X^* with w^*-topology). So for every w^*-open subset $U\subseteq X^*$, the set

$$\partial\varphi^+(U) \stackrel{\mathrm{d}f}{=} \{x\in X: \partial\varphi(x)\subseteq U\}$$

is strongly open. In particular, the graph of $\partial\varphi$, i.e.

$$\mathrm{Gr}\,\partial\varphi=\{(x,x^*)\in X\times X^*: x^*\in\partial\varphi(x)\}$$

is sequentially closed in $X\times X^*_{w^*}$ (see Ref. [10, p. 43]). A point $x\in X$ is a critical point of the locally Lipschitz function φ, if $0\in\partial\varphi(x)$. If $x\in X$ is a critical point, the value $c=\varphi(x)$ is a critical value of φ. It is easy to check that if $x \in X$ is a local extremum of φ (i.e. a local minimum or a local maximum), then $0 \in \partial\varphi(x)$ (i.e. $x \in X$ is a critical point). For further details on the subdifferential theory of locally Lipschitz functions, we refer to Ref. [9].

In the smooth critical point theory, a basic tool in the derivation of minimax characterizations of the critical values, is a compactness condition, known as the Palais–Smale condition. In the present nonsmooth setting this condition takes the following form (see Ref. [11]):

A locally Lipschitz function φ: $X \to \mathbb{R}$ satisfies the *nonsmooth Palais–Smale condition at level* $c \in \mathbb{R}$ (*nonsmooth* PS_c*-condition* for short), if any sequence $\{x_n\}_{n \geqslant 1} \subseteq X$ such that

$$\varphi(x_n) \longrightarrow c \quad \text{and} \quad m^{\varphi}(x_n) \longrightarrow 0,$$

where

$$m^{\varphi}(x_n) \stackrel{\mathrm{d}f}{=} \min\{\|x^*\|_{X^*}: x^* \in \partial\varphi(x_n)\},$$

has a strongly convergent subsequence.

Since for $\varphi \in C^1(X)$ we have $\partial\varphi(x) = \{\varphi'(x)\}$, we see that the above definition is an extension of the smooth PS_c-condition.

We shall need the following nonsmooth version of the linking Theorem (see Ref. [12]). Actually the result of Kourogenis and Papageorgiou [12] is more general, but the formulation that follows suffices for our purposes.

THEOREM 2.1. *If X is a reflexive Banach space, $X = \overline{Y} \oplus \widehat{Y}$ with* $\dim \overline{Y} < +\infty$*, φ: $X \longrightarrow \mathbb{R}$ is a locally Lipschitz function, which satisfies the following hypotheses:*

(i) *there exist $r > 0$ and $\beta \in \mathbb{R}$ such that $\varphi(x) \geqslant \beta$ for all $x \in \widehat{Y} \cap \partial B_r$;*
(ii) *there exist $R > r$, $e \in \widehat{Y}$, $\|e\|_X = 1$ and $\alpha < \beta$ such that if*

$$Q = \{x = te + y : y \in \overline{Y},\ \|y\|_X \leqslant R,\ 0 \leqslant t \leqslant R\}$$

and ∂Q is the boundary of Q in $\overline{Y} \oplus \mathbb{R}e$, we have that $\varphi(x) \leqslant \alpha$ for all $x \in \partial Q$;
(iii) *if*

$$\Gamma \stackrel{\mathrm{d}f}{=} \{\gamma \in C(Q; X) : \gamma|_{\partial Q} = id_{\partial Q}\},$$
$$c \stackrel{\mathrm{d}f}{=} \inf_{\gamma \in \Gamma} \max_{x \in Q} \varphi(\gamma(x))$$

and φ satisfies the nonsmooth PS_c-condition,
then $c \geqslant \beta$ and c is a critical value of φ.

Recall that, if $\{\lambda_n\}_{n\geqslant 1}$ are the distinct eigenvalues of $\big(-\Delta, H_0^1(Z)\big)$, then $\lambda_n \longrightarrow +\infty$ and λ_1 is positive, simple and isolated. Also there is an orthonormal basis $\{u_n\}_{n\geqslant 1} \subseteq H_0^1(Z) \cap C^\infty(Z)$ of $L^2(Z)$, which are eigenfunctions corresponding to the eigenvalues $\{\lambda_n\}_{n\geqslant 1}$, i.e.

$$\begin{aligned} &-\Delta u_n(z) = \lambda_n u_n(z) \quad \forall z \in Z, \\ &u_n|_{\partial Z} = 0, \end{aligned}$$

for $n \geqslant 1$.

If the boundary ∂Z of Z is a C^k-manifold (respectively, a C^∞-manifold) then $u_n \in C^k(\overline{Z})$ (respectively, $u_n \in C^\infty(\overline{Z})$). The sequence $\big\{(1/\sqrt{\lambda_n})u_n\big\}_{n\geqslant 1}$ is an orthonormal basis of $H_0^1(Z)$. For every integer $m \geqslant 1$, let $E(\lambda_m)$ be the eigenspace corresponding to the eigenvalue λ_m. We define

$$\overline{H}_m \stackrel{\mathrm{d}f}{=} \bigoplus_{i=1}^{m-1} E(\lambda_i) \quad \text{and} \quad \widehat{H}_m \stackrel{\mathrm{d}f}{=} \bigoplus_{i=m+1}^{\infty} E(\lambda_i).$$

We have the following orthogonal direct sum decomposition:

$$H_0^1(Z) = \overline{H}_m \oplus E(\lambda_m) \oplus \widehat{H}_m. \tag{2.1}$$

The eigenspace $E(\lambda_m) \subseteq H_0^1(Z) \cap C^\infty(Z)$ has the unique continuation property, namely if $u \in E(\lambda_m)$ is such that u vanishes on a set of positive measure, then $u(z) = 0$ for all $z \in Z$.

If we set

$$\overline{V}_m \stackrel{\mathrm{d}f}{=} \overline{H}_m \oplus E(\lambda_m) \quad \text{and} \quad \widehat{W}_m \stackrel{\mathrm{d}f}{=} E(\lambda_m) \oplus \widehat{H}_m,$$

then we have the following variational characterizations of the eigenvalues (the so called Rayleigh quotients):

$$\lambda_1 = \min_{\substack{x \in H_0^1(Z) \\ x \neq 0}} \frac{\|\nabla x\|_2^2}{\|x\|_2^2}$$

and for $m \geqslant 2$, we have

$$\lambda_m = \max_{\substack{v \in \overline{V}_m \\ v \neq 0}} \frac{\|\nabla v\|_2^2}{\|v\|_2^2}, \tag{2.2}$$

where the maximum is attained on $E(\lambda_m)$. Also

$$\lambda_m = \min_{\substack{w \in \widehat{W}_m \\ w \neq 0}} \frac{\|\nabla w\|_2^2}{\|w\|_2^2}, \tag{2.3}$$

where the minimum is attained on $E(\lambda_m)$ and finally, we have

$$\lambda_m = \min_{\substack{Y \subseteq H_0^1(Z) \\ \dim Y = m}} \max_{\substack{y \in Y \\ y \neq 0}} \frac{\|\nabla y\|_2^2}{\|y\|_2^2}. \tag{2.4}$$

3. Main Result

Our hypotheses on the nonsmooth potential j are the following:

H(j) $j: Z \times \mathbb{R} \longrightarrow \mathbb{R}$ is a function, such that

(i) for all $\zeta \in \mathbb{R}$, the function $z \longmapsto j(z, \zeta)$ is measurable;

(ii) for almost all $z \in Z$, the function $\zeta \longmapsto j(z, \zeta)$ is locally Lipschitz and $j(z, 0) = 0$;

(iii) there exists $\eta \in L^\infty(Z)$, such that

$$|u| \leqslant \eta(z) \quad \text{for a.a. } z \in Z, \text{ all } \zeta \in \mathbb{R} \text{ and all } u \in \partial j(z, \zeta);$$

(iv) there exists $m \in \mathbb{N}$, $m \leqslant k$, such that

$$\limsup_{\zeta \to 0} \frac{u}{\zeta} < \lambda_m - \lambda_k,$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z, \zeta)$ and

$$\sup_{u \in \partial j(z,\zeta)} |u| \longrightarrow 0 \quad \text{as } |\zeta| \to +\infty$$

for almost all $z \in Z$;

(v) we have

$$\liminf_{|\zeta| \to +\infty} j(z, \zeta) \geqslant 0,$$

uniformly for almost all $z \in Z$ and

$$j(z, \zeta) \geqslant \tfrac{1}{2}(\lambda_{m-1} - \lambda_k)\zeta^2,$$

uniformly for almost all $z \in Z$, and all $\zeta \in \mathbb{R}$.

Evidently hypothesis H(j)(iii) implies that for almost all $z \in Z$, $j(z, \cdot)$ is globally Lipschitz.

Let φ: $H_0^1(Z) \longrightarrow \mathbb{R}$ be the energy functional defined by

$$\varphi(x) \stackrel{\mathrm{d}f}{=} \frac{1}{2}\|\nabla x\|_2^2 - \frac{\lambda_k}{2}\|x\|_2^2 - \int_Z j\big(z, x(z)\big)\,\mathrm{d}z \quad \forall x \in H_0^1(Z).$$

We know that φ is locally Lipschitz (see e.g. Ref. [13, p. 313]).

THEOREM 3.1. *If hypotheses* H(j) *hold, then* φ *satisfies the nonsmooth* PS_c *condition for* $c > 0$.

Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ be a sequence, such that

$$\varphi(x_n) \longrightarrow c > 0 \quad \text{and} \quad m^{\varphi}(x_n) \longrightarrow 0. \tag{3.1}$$

Let $x_n^* \in \partial\varphi(x_n)$ be such that

$$\|x_n^*\|_{H^{-1}(Z)} = m^{\varphi}(x_n) \quad \forall n \geqslant 1. \tag{3.2}$$

The existence of such elements follows from the weak compactness of sets $\partial\varphi(x_n) \subseteq H_0^1(Z)$ and the weak lower semicontinuity of the norm functional in Banach spaces. We have

$$x_n^* = Ax_n - \lambda_k x_n - u_n^*, \tag{3.3}$$

with $A \in \mathcal{L}\big(H_0^1(Z), H^{-1}(Z)\big)$ being the operator defined by

$$\langle Ax, y\rangle_{H_0^1(Z)} \stackrel{\mathrm{d}f}{=} \int_Z \big(\nabla x(z), \nabla y(z)\big)_{\mathbb{R}^N}\mathrm{d}z \quad \forall x, y \in H_0^1(Z)$$

and $u_n^* \in L^2(Z)$, with $u_n^*(z) \in \partial j\big(z, x_n(z)\big)$ for almost all $z \in Z$ (see Ref. [10, p. 83]). Evidently $A \geqslant 0$ and so A is maximal monotone.

Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the sequence of distinct eigenvalues of $\big(-\Delta, H_0^1(Z)\big)$ and let $E(\lambda_i)$ be the eigenspace corresponding to the eigenvalue λ_i for $i \geqslant 1$. From (2.1), for every $n \geqslant 1$, we can write that

$$x_n = v_n + x_n^0 + w_n \quad \text{with } v_n \in \overline{H}_k, \quad x_n^0 \in E(\lambda_k), \quad w_n \in \widehat{H}_k.$$

From the parallelogram identity and the orthogonality relations, we see that

$$\|v_n + w_n\|_{H_0^1(Z)} = \|v_n - w_n\|_{H_0^1(Z)} \quad \forall n \geqslant 1. \tag{3.4}$$

Also from the variational characterization of the eigenvalues $\{\lambda_i\}_{i\geqslant 1}$, we have

$$\|\nabla w_n\|_2^2 \geqslant \lambda_{k+1}\|w_n\|_2^2 \quad \forall n \geqslant 1 \tag{3.5}$$

and

$$\|\nabla v_n\|_2^2 \leqslant \lambda_{k-1}\|v_n\|_2^2 \quad \forall n \geqslant 1. \tag{3.6}$$

From (3.3)–(3.6), we have

$$\begin{aligned}
\langle x_n^*, w_n - v_n\rangle_{H_0^1(Z)} &= \langle Ax_n - \lambda_k x_n - u_n^*, w_n - v_n\rangle_{H_0^1(Z)} \\
&= \|\nabla w_n\|_2^2 - \|\nabla v_n\|_2^2 - \lambda_k\|w_n\|_2^2 + \lambda_k\|v_n\|_2^2 \\
&\quad - \int_Z u_n^*(z)\big(w_n - v_n\big)(z)\mathrm{d}z \\
&\geqslant \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|\nabla w_n\|_2^2 - \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|\nabla v_n\|_2^2 \\
&\quad - \int_Z u_n^*(z)\big(w_n - v_n\big)(z)\mathrm{d}z.
\end{aligned}$$

Thus from hypothesis $\mathrm{H}(j)$(iii), we obtain

$$\begin{aligned}
&\|x_n^*\|_{H^{-1}(Z)}\|w_n - v_n\|_{H_0^1(Z)} \\
&\quad \geqslant \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|\nabla w_n\|_2^2 - \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|\nabla v_n\|_2^2 - c_1\|w_n - v_n\|_{H_0^1(Z)},
\end{aligned}$$

for some $c_1 > 0$ and from (3.4), also

$$\begin{aligned}
&\|x_n^*\|_{H^{-1}(Z)}\|w_n + v_n\|_{H_0^1(Z)} \\
&\quad \geqslant \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|\nabla w_n\|_2^2 - \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|\nabla v_n\|_2^2 - c_1\|w_n + v_n\|_{H_0^1(Z)}.
\end{aligned}$$

Using also Poincaré's inequality, we have

$$\begin{aligned}
&\big(c_1 + \|x_n^*\|_{H^{-1}(Z)}\big)\|w_n + v_n\|_{H_0^1(Z)} \\
&\quad \geqslant \left(1 - \frac{\lambda_k}{\lambda_{k+1}}\right)\|\nabla w_n\|_2^2 - \left(1 - \frac{\lambda_k}{\lambda_{k-1}}\right)\|\nabla v_n\|_2^2 \\
&\quad \geqslant c_2\left(\|\nabla w_n\|_2^2 + \|\nabla v_n\|_2^2\right) \geqslant c_3\|w_n + v_n\|_{H_0^1(Z)}^2,
\end{aligned}$$

for some $c_2, c_3 > 0$. Thus the sequence $\{w_n + v_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ is bounded.

So passing to a subsequence, we may assume that

$$w_n + v_n \longrightarrow h \quad \text{weakly in } H_0^1(Z) \tag{3.7}$$

and from the compactness of the embedding $H_0^1(Z) \subseteq L^2(Z)$, also

$$w_n + v_n \longrightarrow h \quad \text{in } L^2(Z). \tag{3.8}$$

Then, from (3.2) and (3.1), we have

$$\begin{aligned}\langle Ax_n - \lambda_k x_n - u_n^*, w_n + v_n - h\rangle &= \langle x_n^*, w_n + v_n - h\rangle_{H_0^1(Z)} \\ &\leqslant \varepsilon_n \| w_n + v_n - h\|_{H_0^1(Z)},\end{aligned} \tag{3.9}$$

for some $\varepsilon_n \searrow 0$. Exploiting the orthogonality relations, we have

$$\begin{aligned}\langle \lambda_k x_n, w_n + v_n - h\rangle_{H_0^1(Z)} &= \int_Z \lambda_k x_n(z)\big(w_n + v_n - h\big)(z)\mathrm{d}z \\ &= \int_Z \lambda_k\big(w_n+v_n\big)(z)\big(w_n+v_n-h\big)(z)\mathrm{d}z \longrightarrow 0.\end{aligned} \tag{3.10}$$

Also because the sequence $\{u_n^*\}_{n\geqslant 1} \subseteq L^2(Z)$ is bounded, from (3.8) we have that

$$\langle u_n^*, w_n + v_n - h\rangle_{H_0^1(Z)} = \int_Z u_n^*(z)\big(w_n + v_n - h\big)(z)\mathrm{d}z \longrightarrow 0. \tag{3.11}$$

Passing to the limit as $n \to +\infty$ in (3.9) and using (3.10) and (3.11), we obtain

$$\limsup_{n\to+\infty} \langle Ax_n, w_n + v_n - h\rangle_{H_0^1(Z)} \leqslant 0,$$

thus

$$\limsup_{n\to+\infty} \langle A(w_n + v_n), w_n + v_n - h\rangle_{H_0^1(Z)} \leqslant 0,$$

so, by the maximal monotonicity of A, we have

$$\langle A(w_n + v_n), w_n + v_n\rangle_{H_0^1(Z)} \longrightarrow \langle Ah, h\rangle_{H_0^1(Z)}$$

and finally

$$\|\nabla(w_n + v_n)\|_2 \longrightarrow \|\nabla h\|_2.$$

Recalling that

$$\nabla(w_n+v_n)\longrightarrow\nabla h \quad \text{weakly in } L^2(Z;\mathbb{R}^N)$$

(see (3.7)), from the Kadec–Klee property of Hilbert spaces, it follows that

$$\nabla(w_n+v_n)\longrightarrow\nabla h \quad \text{in } L^2(Z;\mathbb{R}^N)$$

and thus, we have

$$w_n+v_n\longrightarrow h \quad \text{in } H_0^1(Z). \tag{3.12}$$

Next we claim that the sequence $\{x_n^0\}\subseteq E(\lambda_k)\subseteq H_0^1(Z)$ is bounded. Suppose that this is not the case. Then by passing to a subsequence if necessary, we may assume that

$$\mu_n \stackrel{\mathrm{d}f}{=} \|x_n^0\|_{H_0^1(Z)}\longrightarrow+\infty.$$

Let us set

$$y_n^0 \stackrel{\mathrm{d}f}{=} \frac{x_n^0}{\mu_n} \quad \forall n\geqslant 1.$$

Because $E(\lambda_k)$ is finite dimensional (and so all norms are equivalent), we may assume that

$$y_n^0\longrightarrow y^0, \quad \text{in } C(Z),$$

for some $y^0\neq 0$ and by the unique continuation property of the eigenfunctions of $\big(-\Delta, H_0^1(Z)\big)$, we have that $y^0(z)\neq 0$ for almost all $z\in Z$. We have

$$u_n^*(z)\in\partial j\big(z, w_n(z)+v_n(z)+\mu_n y_n^0(z)\big) \quad \text{for a.a. } z\in Z,$$

with $\mu_n\to+\infty$. From (3.12), by passing to a subsequence if necessary, we may assume that

$$\begin{aligned}
&w_n+v_n\longrightarrow h \quad \text{in } L^2(Z),\\
&w_n(z)+v_n(z)\longrightarrow h(z) \quad \text{for a.a. } z\in Z,\\
&\big|w_n(z)+v_n(z)\big|\leqslant\vartheta(z) \quad \text{for a.a. } z\in Z,
\end{aligned}$$

with $\vartheta\in L^2(Z)$. We have

$$\begin{aligned}
\big|x_n(z)\big| &= \big|w_n(z)+v_n(z)+\mu_n y_n^0(z)\big|\\
&\geqslant \mu_n\big|y_n^0(z)\big|-\vartheta(z)\longrightarrow+\infty \quad \text{for a.a. } z\in Z.
\end{aligned} \tag{3.13}$$

From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$, we have

$$x_n^* = Ax_n - \lambda_k x_n - u_n^* \longrightarrow 0 \quad \text{in } H^{-1}(Z).$$

Since

$$Au^0 = \lambda_k u^0 \quad \forall u^0 \in E(\lambda_k)$$

for every $n \geqslant 1$, we have

$$\begin{aligned} x_n^* &= Ax_n - \lambda_k x_n - u_n^* \\ &= A(w_n + v_n) - \lambda_k(w_n + v_n) + Ax_n^0 - \lambda_k x_n^0 - u_n^* \\ &= A(w_n + v_n) - \lambda_k(w_n + v_n) - u_n^*. \end{aligned} \tag{3.14}$$

From (3.7) and the fact that $A \in \mathcal{L}\big(H_0^1(Z), H^{-1}(Z)\big)$, we have

$$A(w_n + v_n) \longrightarrow Ah \quad \text{weakly in } H^{-1}(Z)$$

and from (3.8), we have

$$\lambda_k(w_n + v_n) \longrightarrow \lambda_k h \quad \text{in } L^2(Z).$$

Moreover, from the second part of hypothesis H(j)(iv) and (3.13), we know that

$$u_n^*(z) \longrightarrow 0 \quad \text{for a.a. } z \in Z$$

and by the Lebesgue dominated convergence theorem, it follows that

$$u_n^* \longrightarrow 0 \quad \text{in } L^2(Z).$$

So, if we pass to the limit in (3.14), we obtain

$$0 = Ah - \lambda_k h,$$

so

$$Ah = \lambda_k h,$$

thus

$$\begin{aligned} &-\Delta h(z) = \lambda_k h(z) \quad \text{for a.a. } z \in Z, \\ &h|_{\partial Z} = 0. \end{aligned}$$

But $h \in E(\lambda_k)^{\perp}$, so it follows that $h=0$ and

$$w_n + v_n \longrightarrow 0 \quad \text{in } H_0^1(Z), \tag{3.15}$$

Again from the choice of the sequence $\{x_n\} \subseteq H_0^1(Z)$, using (3.15) we have

$$\begin{aligned}
0 < c &= \lim_{n\to+\infty} \varphi(x_n) \\
&= \lim_{n\to+\infty} \left[\frac{1}{2}\|\nabla x_n\|_2^2 - \frac{\lambda_k}{2}\|x_n\|_2^2 - \int_Z j\big(z, x_n(z)\big)\mathrm{d}z \right] \\
&= \lim_{n\to+\infty} \left[\frac{1}{2}\|\nabla (w_n+v_n)\|_2^2 - \frac{\lambda_k}{2}\|w_n+v_n\|_2^2 - \int_Z j\big(z, x_n(z)\big)\mathrm{d}z \right] \\
&\leqslant \limsup_{n\to+\infty} \left(-\int_Z j\big(z, x_n(z)\big)\mathrm{d}z \right).
\end{aligned} \tag{3.16}$$

On the other hand by (3.13), Fatou's lemma and the first part of hypothesis H(j)(v), we have

$$\limsup_{n\to+\infty} \left(-\int_Z j\big(z, x_n(z)\big)\mathrm{d}z \right) \leqslant -\int_Z \liminf_{n\to+\infty} j\big(z, x_n(z)\big)\mathrm{d}z \leqslant 0. \tag{3.17}$$

From (3.16) and (3.17), we obtain a contradiction. So indeed the sequence $\{x_n^0\}_{n\geqslant 1} \subseteq E(\lambda_k) \subseteq H_0^1(Z)$ is bounded. From this it follows that the sequence $\{x_n\}_{n\geqslant 1} \subseteq H_0^1(Z)$ is bounded. Thus we may assume that

$$\begin{aligned}
&x_n \longrightarrow x \quad \text{weakly in } H_0^1(Z), \\
&x_n \longrightarrow x \quad \text{in } L^2(Z).
\end{aligned}$$

From (3.13), we have that

$$\langle x_n^*, x_n - x\rangle_{H_0^1(Z)} = \langle Ax_n - \lambda_k x_n - u_n^*, x_n - x\rangle_{H_0^1(Z)} \leqslant \varepsilon_n \|x_n - x\|_{H_0^1(Z)},$$

with $\varepsilon_n \searrow 0$. Since, from hypothesis H(j)(iii), we have

$$\langle u_n^*, x_n - x\rangle_{H_0^1(Z)} = \int_Z u_n^*(z)\big(x_n - x\big)(z)\mathrm{d}z \longrightarrow 0$$

and

$$\langle \lambda_k x_n, x_n - x\rangle_{H_0^1(Z)} = \int_Z \lambda_k x_n(z)\big(x_n - x\big)(z)\mathrm{d}z \longrightarrow 0$$

so also

$$\limsup_{n\to+\infty} \langle Ax_n, x_n - x\rangle_{H_0^1(Z)} \leqslant 0.$$

As A is maximal monotone, it is also generalized pseudomonotone, so

$$\langle Ax_n, x_n\rangle \longrightarrow \langle Ax, x\rangle,$$

and

$$\|\nabla x_n\|_2 \longrightarrow \|\nabla x\|_2$$

and finally

$$x_n \longrightarrow x \quad \text{in } H_0^1(Z). \qquad \square$$

Now we can formulate our main theorem and establish the existence of non-trivial solutions for problem (1.1). By a solution of (1.1) we understand a function $x \in H_0^1(Z)$ such that it satisfies the equation pointwise for almost all $z \in Z$. Evidently such a solution belongs in $H^2(Z)$. In fact standard linear elliptic regularity theory implies that $x \in C^1(\overline{Z})$.

THEOREM 3.2. *If hypotheses* H*(j) hold, then problem* (1.1) *has at least one nontrivial solution.*

Proof. By virtue of the first part of hypothesis $\bar{H}(j)$(iv), we can find $\xi < \lambda_m - \lambda_k \leqslant 0$ and $\delta > 0$ such that

$$\frac{u}{\zeta} \leqslant \xi \quad \text{for a.a. } z \in Z, \text{ all } |\zeta| \leqslant \delta \quad \text{and all } u \in \partial j(z,\zeta). \tag{3.18}$$

Also from hypothesis $H(j)$(iii), we have

$$\left|\frac{u}{\zeta}\right| \leqslant \eta_1(z) \quad \text{for a.a. } z \in Z, \text{ all } |\zeta| \geqslant \delta \quad \text{and all } u \in \partial j(z,\zeta),$$

for some $\eta_1 \in L^\infty(Z)$. Therefore we can find $\xi_1 > 0$ and $i > k$ such that

$$\frac{u}{\zeta} \leqslant \xi_1 \leqslant \tfrac{1}{3}(\lambda_i - \lambda_k) \quad \text{for a.a. } z \in Z, \text{ all } \zeta \neq 0 \text{ and all } u \in \partial j(z,\zeta) \tag{3.19}$$

and

$$\frac{32\xi^2}{\lambda_m - \lambda_k - \xi} + \lambda_k + 3\xi_1 < \lambda_i. \tag{3.20}$$

Let

$$\overline{H}_{m,i} \stackrel{\mathrm{df}}{=} \bigoplus_{j=m}^{i-1} E(\lambda_j) \quad \text{and} \quad \widehat{W}_i \stackrel{\mathrm{df}}{=} \bigoplus_{j \geqslant i} E(\lambda_j).$$

Let $w \in \widehat{W}_i$ and $y \in \overline{H}_{m,i}$. Using the variational expressions for the eigenvalues of $\big(-\Delta, H_0^1(Z)\big)$, we have

$$\begin{aligned}
\varphi(w+y) &= \frac{1}{2}\|\nabla(w+y)\|_2^2 - \frac{\lambda_k}{2}\|w+y\|_2^2 - \int_Z j\big(z,(w+y)(z)\big)\mathrm{d}z \\
&= \frac{1}{2}\|\nabla w\|_2^2 + \frac{1}{2}\|\nabla y\|_2^2 - \frac{\lambda_k}{2}\|w\|_2^2 - \frac{\lambda_k}{2}\|y\|_2^2 \\
&\quad - \int_Z j\big(z,(w+y)(z)\big)\mathrm{d}z \\
&\geqslant \frac{1}{4}\|\nabla w\|_2^2 + \frac{1}{4}\|\nabla y\|_2^2 - \frac{\lambda_k}{2}\|w\|_2^2 - \frac{\lambda_k}{2}\|y\|_2^2 \\
&\quad + \frac{\lambda_i}{4}\|w\|_2^2 + \frac{\lambda_m}{4}\|y\|_2^2 - \int_Z j\big(z,(w+y)(z)\big)\mathrm{d}z. \qquad (3.21)
\end{aligned}$$

For $z \in Z \setminus N$ with $|N| = 0$, we consider two cases:

Case 1. $|w(z) + y(z)| > \delta$.
First suppose that $w(z) + y(z) > \delta$. Invoking the Lebourg mean value theorem (see Ref. [10, p. 41] or Lebourg [2]), we obtain

$$j\big(z,(w+y)(z)\big) - j(z,\delta) = u_t^*\big[(w+y)(z) - \delta\big]$$

and

$$j(z,\delta) = j(z,\delta) - j(z,0) = v_t^*\delta,$$

where

$$u_t^* \in \partial j\big(z, t(z)(w+y)(z) + (1-t(z))\delta\big), \quad 0 < t(z) < 1$$

and

$$v_t^* \in \partial j\big(z, t_1(z)\delta\big), \quad 0 < t_1(z) < 1.$$

Then, using (3.19), we have

$$u_t^* \leqslant \xi_1\big[t(z)(w+y)(z) + (1-t(z))\delta\big] \leqslant \xi_1(w+y)(z)$$

and using (3.18), we have

$$v_t^* \leqslant \xi t_1(z)\delta \leqslant \xi\delta.$$

So it follows that

$$j(z,\delta) \leqslant \xi\delta^2$$

and

$$\begin{aligned} j\big(z,(w+y)(z)\big) &\leqslant \xi_1(w+y)(z)\big[(w+y)(z)-\delta\big]+\xi\delta^2 \\ &\leqslant \xi_1(w+y)^2(z)-\xi_1\delta^2+\xi\delta^2 \\ &= \xi_1(w+y)^2(z)-(\xi_1-\xi)\delta^2. \end{aligned} \tag{3.22}$$

Next suppose that $w(z)+y(z)<-\delta$. Again via the Lebourg mean value theorem, we have

$$j\big(z,(w+y)(z)\big)-j(z,-\delta)=u_t^*\big((w+y)(z)+\delta\big),$$

and

$$j(z,-\delta)=j(z,-\delta)-j(z,0)=v_t^*(-\delta).$$

where

$$v_t^*\in\partial j\big(z,t_1(z)(-\delta)\big),\quad 0<t_1(z)<1$$

and

$$u_t^*\in\partial j\big(z,t(z)(w+y)(z)+(1-t(z))(-\delta)\big),\quad 0<t(z)<1.$$

In this case, we have

$$u_t^*\geqslant\xi_1(w+y)(z)\quad\text{and}\quad v_t^*\geqslant\xi(-\delta)$$

and so

$$j(z,\delta)\leqslant\xi\delta^2$$

and

$$j\big(z,(w+y)(z)\big)\ \leqslant\ \xi_1(w+y)^2(z)-(\xi_1-\xi)\delta^2.$$

The last inequality is the same as (3.22). So (3.22) holds when $\big|(w+y)(z)\big|>\delta$. Then, for $\big|(w+y)(z)\big|>\delta$, we can write

$$\begin{aligned} &\frac{\lambda_i-\lambda_k+\xi_1}{4}w(z)^2+\frac{\lambda_m-\lambda_k+\xi}{4}y(z)^2-j\big(z,(w+y)(z)\big) \\ &\quad\geqslant\frac{\lambda_i-\lambda_k+\xi_1}{4}w(z)^2+\frac{\lambda_m-\lambda_k+\xi}{4}y(z)^2-\xi_1(w+y)^2(z)+(\xi_1-\xi)\delta^2 \\ &\quad=\frac{\lambda_i-\lambda_k+\xi_1}{4}w(z)^2+\frac{\lambda_m-\lambda_k+\xi}{4}y(z)^2-\xi y(z)^2 \\ &\qquad-(\xi_1-\xi)y(z)^2-\xi_1w(z)^2-2\xi_1\big(wy\big)(z)+(\xi_1-\xi)\delta^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda_i - \lambda_k - 3\xi_1}{4} w(z)^2 + \frac{\lambda_m - \lambda_k - \xi}{4} y(z)^2 - \frac{\xi}{2} y(z)^2 \\
&\quad -(\xi_1 - \xi) y(z)^2 - 2\xi_1 \big(wy\big)(z) + (\xi_1 - \xi)\delta^2 \\
&= \frac{\lambda_i - \lambda_k - 3\xi_1}{8} w(z)^2 + \frac{\lambda_m - \lambda_k - \xi}{4} y(z)^2 - \frac{\xi}{2} y(z)^2 \\
&\quad + \frac{\lambda_i - \lambda_k - 3\xi_1}{8} w(z)^2 - (\xi_1 - \xi) y(z)^2 - 2\xi_1 \big(wy\big)(z) + (\xi_1 - \xi)\delta^2 \\
&\geqslant \frac{\lambda_i - \lambda_k - 3\xi_1}{8} w(z)^2 + \frac{\lambda_m - \lambda_k - \xi}{4} y(z)^2 - 2\xi \big(wy\big)(z) \\
&\quad + \frac{\lambda_i - \lambda_k - 3\xi_1}{8} w(z)^2 - (\xi_1 - \xi) y(z)^2 \\
&\quad -2(\xi_1 - \xi)\big(wy\big)(z) + (\xi_1 - \xi)\delta^2
\end{aligned} \tag{3.23}
$$

(since $\xi \leqslant 0$).

Case 2. $\big|w(z) + y(z)\big| \leqslant \delta$.

In this case, using the Lebourg mean value theorem, (3.18) and the facts that $\xi < 0 \leqslant \xi_1$, $\lambda_i - \lambda_k > 3\xi_1$, we have

$$
\begin{aligned}
&\frac{\lambda_i - \lambda_k + \xi_1}{4} w(z)^2 + \frac{\lambda_m - \lambda_k + \xi}{4} y(z)^2 - j\big(z, (w+y)(z)\big) \\
&\geqslant \frac{\lambda_i - \lambda_k + \xi_1}{4} w(z)^2 + \frac{\lambda_m - \lambda_k + \xi}{4} y(z)^2 - \xi (w+y)^2(z) \\
&\geqslant \frac{\lambda_i - \lambda_k + \xi_1}{4} w(z)^2 + \frac{\lambda_m - \lambda_k + \xi}{4} y(z)^2 - \xi_1 w(z)^2 - \xi y(z)^2 - 2\xi (wy)(z) \\
&\geqslant \frac{\lambda_i - \lambda_k - 3\xi_1}{4} w(z)^2 + \frac{\lambda_m - \lambda_k - \xi}{4} y(z)^2 - \frac{\xi}{2} y(z)^2 - 2\xi (wy)(z) \\
&\geqslant \frac{\lambda_i - \lambda_k - 3\xi_1}{8} w(z)^2 + \frac{\lambda_m - \lambda_k - \xi}{4} y(z)^2 - 2\xi (wy)(z).
\end{aligned} \tag{3.24}
$$

Note that on the right-hand side of (3.23) the first three summands are the same as on the right-hand side of (3.24).

Now, we can find $\mu > 0$ such that for every $\sigma \in \mathbb{R}$ and every $\tau \in [-\mu, \mu]$, we have

$$
\big(\lambda_i - \lambda_k - 3\xi_1\big)\sigma^2 - 8(\xi_1 - \xi)\tau^2 - 16(\xi_1 - \xi)\sigma\tau + (\xi_1 - \xi)\delta^2 \geqslant 0. \tag{3.25}
$$

Because $\overline{H}_{m,i}$ is finite dimensional, all norms are equivalent on $\overline{H}_{m,i}$ and so we can find $\varrho > 0$, such that, if $\|y\|_{H_0^1(Z)} \leqslant \varrho$, $y \in \overline{H}_{m,i}$, then $\big|y(z)\big| \leqslant \mu$ for all $z \in Z$. Then, for $\|y\|_{H_0^1(Z)} \leqslant \varrho$, $y \in \overline{H}_{m,i}$ and $w \in \widehat{W}_i$, from (3.21), (3.23)–(3.25), (3.20) and the fact that $\xi < \lambda_m - \lambda_k$, we have

$$
\begin{aligned}
\varphi(w+y) \geqslant\ & \frac{1}{4}\|\nabla w\|_2^2+\frac{1}{4}\|\nabla y\|_2^2-\frac{\lambda_k}{2}\|w\|_2^2-\frac{\lambda_k}{2}\|y\|_2^2 \\
& +\frac{\lambda_i}{4}\|w\|_2^2+\frac{\lambda_m}{4}\|y\|_2^2-\int_Z j\big(z,(w+y)(z)\big)\mathrm{d}z \\
=\ & \left[\frac{1}{4}\|\nabla w\|_2^2+\frac{1}{4}\|\nabla y\|_2^2-\frac{\xi_1}{4}\|w\|_2^2-\frac{\xi}{4}\|y\|_2^2-\frac{\lambda_K}{4}\|w\|_2^2-\frac{\lambda_k}{4}\|y\|_2^2\right] \\
& +\frac{\lambda_i-\lambda_k+\xi_1}{4}\|w\|_2^2+\frac{\lambda_m-\lambda_k+\xi}{4}\|y\|_2^2-\int_Z j\big(z,(w+y)(z)\big)\mathrm{d}z \\
=\ & \left[\frac{1}{4}\|\nabla w\|_2^2+\frac{1}{4}\|\nabla y\|_2^2-\frac{\xi_1}{4}\|w\|_2^2-\frac{\xi}{4}\|y\|_2^2-\frac{\lambda_k}{4}\|w\|_2^2-\frac{\lambda_k}{4}\|y\|_2^2\right] \\
& +\ \frac{\lambda_i-\lambda_k-3\xi_1}{8}w(z)^2+\frac{\lambda_m-\lambda_k-\xi}{4}y(z)^2-2\xi(wy)(z) \\
\geqslant\ & \frac{1}{4}\|\nabla w\|_2^2+\frac{1}{4}\|\nabla y\|_2^2-\frac{\xi_1}{4}\|w\|_2^2-\frac{\xi}{4}\|y\|_2^2-\frac{\lambda_k}{4}\|w\|_2^2-\frac{\lambda_k}{4}\|y\|_2^2 \\
\geqslant\ & \frac{1}{4}\left(1-\frac{\lambda_k}{\lambda_i}-\frac{\xi_1}{\lambda_i}\right)\|\nabla w\|_2^2+\frac{1}{4}\left(1-\frac{\lambda_k}{\lambda_m}-\frac{\xi}{\lambda_m}\right)\|\nabla y\|_2^2>0. \quad (3.26)
\end{aligned}
$$

Next we consider

$$
\begin{aligned}
\overline{H}_m &= \bigoplus_{j=1}^{m-1} E(\lambda_j), \\
\widehat{W}_m &= \bigoplus_{j\geqslant m} E(\lambda_j) = \overline{H}_{m,i}\oplus \widehat{W}_i, \\
\overline{V}_m &= \bigoplus_{j=1}^{m} E(\lambda_j).
\end{aligned}
$$

From (3.26), we see that we can find $r>0$ and $\beta>0$ such that

$$\inf\{\varphi(x): x\in \widehat{W}_m,\ \|x\|_{H_0^1(Z)}=r\} = \beta > 0.$$

We claim that for any $0<\alpha<\beta$, we can find $R>r$ large enough, such that if $u\in\overline{V}_m$ with $\|u\|_{H_0^1(Z)}=R$, then $\varphi(u)\leqslant\alpha$. Indeed if this is not the case, we can find a sequence $\{u_n\}_{n\geqslant 1}\subseteq\overline{V}_m$ with $\|u_n\|_{H_0^1(Z)}\longrightarrow+\infty$ and $\varepsilon>0$, such that

$$\varphi(u_n)\geqslant\varepsilon \quad \forall n\geqslant 1,$$

thus

$$\liminf_{n\to+\infty}\varphi(u_n)\geqslant\varepsilon,$$

so

$$\liminf_{n\to+\infty}\left[\frac{1}{2}\|\nabla u_n\|_2^2-\frac{\lambda_k}{2}\|u_n\|_2^2-\int_Z j\big(z,u_n(z)\big)\mathrm{d}z\right]\geqslant\varepsilon.$$

Since $m\leqslant k$ and $(1/2)\|\nabla u_n\|_2^2\leqslant(\lambda_k/2)\|u_n\|_2^2$, we have

$$\liminf_{n\to+\infty}\left(-\int_Z j\big(z,u_n(z)\big)\mathrm{d}z\right)\geqslant\varepsilon.$$

Let

$$\widehat{e}_n\stackrel{\mathrm{d}f}{=}\frac{u_n}{\|u_n\|_{H_0^1(Z)}}\quad\forall n\geqslant 1.$$

Then

$$\widehat{e}_n\longrightarrow\widehat{e}\quad\text{in } H_0^1(Z),$$

for some $\widehat{e}\in\overline{V}_m$ (since $\overline{V}_m$ is finite dimensional). Moreover, from the unique continuation property $e(z)\neq 0$ for almost all $z\in Z$ and so

$$\big|u_n(z)\big|\longrightarrow+\infty\quad\text{for a.a. } z\in Z.$$

So from Fatou's lemma and the first part of hypothesis $H(j)(v)$, we have

$$\begin{aligned}\varepsilon&\leqslant\liminf_{n\to+\infty}\left(-\int_Z j\big(z,u_n(z)\big)\mathrm{d}z\right)=-\limsup_{n\to+\infty}\int_Z j\big(z,u_n(z)\big)\mathrm{d}z\\&\leqslant-\liminf_{n\to+\infty}\int_Z j\big(z,u_n(z)\big)\mathrm{d}z\leqslant-\int_Z\liminf_{n\to+\infty}j\big(z,u_n(z)\big)\mathrm{d}z\leqslant 0,\end{aligned}\tag{3.27}$$

a contradiction.

In addition, using the second part of hypothesis H(j)(v), for any $y\in\overline{H}_m$, we have

$$\begin{aligned}\varphi(y)&=\frac{1}{2}\|\nabla y\|_2^2-\frac{\lambda_k}{2}\|y\|_2^2-\int_Z j\big(z,y(z)\big)\mathrm{d}z\\&\leqslant\frac{1}{2}\|\nabla y\|_2^2-\frac{\lambda_k}{2}\|y\|_2^2+\frac{\lambda_k}{2}\|y\|_2^2-\frac{\lambda_{m-1}}{2}\|y\|_2^2\\&=\frac{1}{2}\|\nabla y\|_2^2-\frac{\lambda_{m-1}}{2}\|y\|_2^2\ \leqslant\ 0,\end{aligned}$$

i.e. $\varphi|_{\overline{H}_m}\leqslant 0$.

Therefore, let $e \in E(\lambda_m)$ be such that $\|e\|_{H_0^1(Z)} = 1$ and let $R > r$ be large enough so that $\varphi(u) \leqslant \alpha$ for all $u \in \overline{V}_m$, $\|u\|_{H_0^1(Z)} = R$. Recall that

$$\alpha < \beta = \inf\{\varphi(x) : x \in \widehat{W}_m,\ \|x\|_{H_0^1(Z)} = r\}.$$

So we can apply Theorem 2.1 with $\overline{Y} = \overline{H}_m$ and $\widehat{Y} = \widehat{W}_m$ and obtain $x_0 \in H_0^1(Z)$, such that $0 \in \partial\varphi(x_0)$ and $\varphi(x_0) \geqslant \beta > 0 = \varphi(0)$, so $x_0 \neq 0$. If follows easily that x_0 is the desired nontrivial solution of (1.1). □

Remark 3.3. In the proof of Theorem 3.1 (see (3.17)) as well as in the proof of Theorem 3.2 (see (3.27)), the application of Fatou's lemma is permitted since by the first part of hypothesis H(j)(v), we can find $M > 0$ such that

$$j(z, \zeta) \geqslant -1 \quad \text{for a.a. } z \in Z \text{ and all } |\zeta| > M.$$

By hypothesis H(j)(iii) and the Lebourg mean value theorem, we have

$$\left|j(z, \zeta)\right| \leqslant \eta_1(z) \quad \text{for a.a. } z \in Z \quad \text{and all } |\zeta| \leqslant M,$$

with $\eta_1 \in L^\infty(Z)$. So finally

$$j(z, \zeta) \geqslant -\eta_2(z) \quad \text{for a.a. } z \in Z \quad \text{and all } \zeta \in \mathbb{R},$$

with $\eta_2 \in L^\infty(Z)_+$. This permits the use of Fatou's lemma.

Remark 3.4. As a simple example of a superpotential j satisfying hypotheses H(j), we can take the following function (for simplicity we drop the $z \in Z$ dependence; see Figure 1):

$$j(\zeta) \stackrel{\mathrm{d}f}{=} \begin{cases} \dfrac{a}{\zeta} & \text{if } \zeta \leqslant -1, \\ -a\zeta^2 & \text{if } -1 < \zeta \leqslant 1, \\ -\dfrac{a}{\zeta} & \text{if } 1 < \zeta \end{cases}$$

for some $0 < (\lambda_k - \lambda_m)/2 < a \leqslant (\lambda_k - \lambda_{m-1})/2$.

Then we have (see Figure 2):

$$\partial j(\zeta) \stackrel{\mathrm{d}f}{=} \begin{cases} -\dfrac{a}{\zeta^2} & \text{if } \zeta < -1, \\ [-a, 2a] & \text{if } \zeta = -1, \\ -2a\zeta & \text{if } -1 < \zeta < 1, \\ [-2a, a] & \text{if } \zeta = 1, \\ \dfrac{a}{\zeta^2} & \text{if } 1 < \zeta. \end{cases}$$

Figure 1.

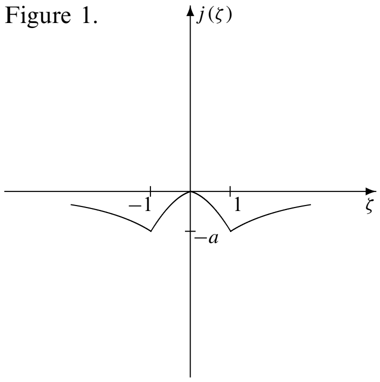
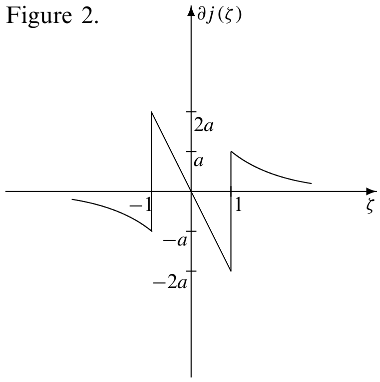


Figure 2.



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